Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Electronic Notes in Theoretical Computer Science 202 (2008) 121–135

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

On the Complexity of Convex Hulls of Subsets of the Two-Dimensional Plane

Ker-I Ko[a,1,2] Fuxiang Yu[a,1,3]

[a] *Department of Computer Science, Stony Brook University, Stony Brook, NY 11794, USA*

Abstract

We investigate the computational complexity of computing the convex hull of a two-dimensional set. We study this problem in the polynomial-time complexity theory of real functions based on the oracle Turing machine model. We show that the convex hull of a two-dimensional Jordan domain S is not necessarily recursively recognizable even if S is polynomial-time recognizable. On the other hand, if the boundary of a Jordan domain S is polynomial-time computable, then the convex hull of S must be NP-recognizable, and it is not necessarily polynomial-time recognizable if $P \neq NP$. We also show that the area of the convex hull of a Jordan domain S with a polynomial-time computable boundary can be computed in polynomial time relative to an oracle function in $\#P$. On the other hand, whether the area itself is a $\#P$ real number depends on the open question of whether $NP = UP$.

Keywords: Convex hulls, two-dimensional set, computational complexity, polynomial time, NP.

1 Introduction

The convex hull of a set S of the two-dimensional plane is the smallest convex set $CH(S)$ that contains S. It is a fundamental concept in mathematics and in computational geometry. For polygons and sets of finite points, there are a number of efficient algorithms to compute their convex hulls (see, for instance, O'Rourke [14] and de Berg et al. [7]). In general, however, no efficient algorithms are known to work for all subsets of the two-dimensional plane. In fact, for some set S, its convex hull could be very complicated and defies a simple algorithm.

In this paper, we study the complexity of computing the convex hull of a given set $S \subseteq \mathbb{R}^2$. In particular, we study two problems about the convex hull $CH(S)$ of a polynomial-time computable set $S \subseteq \mathbb{R}^2$:

[1] This material is based upon work supported by National Science Foundation under grant No. 0430124.

[2] Email: keriko@cs.sunysb.edu

[3] Email: fuxiang@cs.sunysb.edu

doi:10.1016/j.entcs.2008.03.012

Membership Problem: For a polynomial-time computable set S and a given point $\mathbf{z}$, determine whether $\mathbf{z}$ is inside $CH(S)$.

Area Problem: For a polynomial-time computable set S, compute the two-dimensional measure of the convex hull of S.

There are a number of different formulations of the notion of polynomial-time computable sets in the two-dimensional plane. In this paper, we will use three notions introduced in Chou and Ko [3]: polynomial-time computable Jordan domains (i.e., sets whose boundaries are polynomial-time computable Jordan curves), polynomial-time recognizable sets, and strongly polynomial-time recognizable sets. Our main results can be summarized as follows:

(1) There exists a Jordan domain $S \subseteq \mathbb{R}^2$ which is polynomial-time recognizable such that its convex hull is not even recursively recognizable.

(2) If a set $S \subseteq \mathbb{R}^2$ is a Jordan domain and its boundary is polynomial-time computable, or if S is strongly polynomial-time recognizable, then its convex hull $CH(S)$ is strongly nondeterministic polynomial-time recognizable.

(3) If $P \neq NP$, then there exists a Jordan domain $S \subseteq \mathbb{R}^2$ whose boundary is polynomial-time computable such that its convex hull $CH(S)$ is not polynomial-time recognizable.

(4) If a set $S \subseteq \mathbb{R}^2$ is a Jordan domain and its boundary is polynomial-time computable, or if S is strongly polynomial-time recognizable, then the area of its convex hull $CH(S)$ is computable in polynomial-time with the help of an oracle function in $\#P$.

(5) If $FP_1 \neq \#P_1$, then there exists a Jordan domain $S \subseteq \mathbb{R}^2$ whose boundary is polynomial-time computable such that the area of its convex hull $CH(S)$ is not a polynomial-time computable real number.[4]

Our basic computational model for real-valued functions and two-dimensional sets is the oracle Turing machine. For the general theory of computable analysis based on the Turing machine model, see, for instance, Pour-El and Richards [15] and Weihrauch [21]. For the theory of computational complexity of real functions based on this computational model, see Ko [12]. Chou and Ko [3] extended this theory to the study of computational complexity of two-dimensional sets. Computational complexity of problems related to two-dimensional sets has recently been studied in several directions. Rettinger and Weihrauch [17], Rettinger [16], Braverman [1], and Braverman and Yampolsky [2] studied the the computational complexity of Julia sets. Chou and Ko [4] studied the problem of finding paths in a two-dimensional domain. Ko and Yu [13] studied the problem of computing single-valued analytic branches of logarithm and square-root functions on a two-dimensional domain. All these works used Turing machines and oracle Turing machines as the basic model.

[4] FP_1 and $\#P_1$ are functions in FP and $\#P$, respectively, whose inputs are strings from a singleton alphabet.

2 Definitions and Notation

2.1 Discrete complexity classes

In this paper, we will work on both discrete and continuous objects. The basic objects in discrete complexity theory are binary strings $w \in \{0,1\}^*$. We write $\ell(w)$ to denote the length of a string w (and reserve the notation $|x|$ for the absolute value of a real or complex number x).

The fundamental discrete complexity classes we are interested in are the class P of sets accepted by deterministic polynomial-time Turing machines (TMs), and the corresponding function class FP of functions computable by deterministic polynomial-time TMs. In addition to these classes, we are also interested, in this paper, in the following complexity classes (see, e.g., Du and Ko [9] for the formal definitions):

- NP: Sets that are accepted by nondeterministic polynomial-time TMs.
- UP: Sets that are accepted by nondeterministic polynomial-time TMs that have, on any input, at most one accepting computation.
- $\#P$: Functions that compute the number of accepting computations of a nondeterministic polynomial-time TMs.
- $P^{\#P}$: Sets that are accepted by deterministic polynomial-time oracle TMs with the help of an oracle function $f \in \#P$ (we also write $P^{\#P[1]}$ for the sets for which the oracle function $f \in \#P$ is asked at most once during the computation).
- $FP^{\#P}$: Functions that are computable by deterministic polynomial-time oracle TMs with the help of an oracle function $f \in \#P$

The classes NP, UP and $\#P$ have nice characterizations in terms of class P. In the following, we let $\|A\|$ denote the size of a finite set A.

Proposition 2.1 *(a) A set $A \subseteq \{0,1\}^*$ is in NP if and only if there exist a set $B \in P$ and a polynomial function p such that, for any $w \in \{0,1\}^*$,*

$$w \in A \iff (\exists u, \ell(u) = p(\ell(w)))\, \langle w, u\rangle \in B.$$

(b) A set $A \subseteq \{0,1\}^*$ is in UP if and only if there exist a set $B \in P$ and a polynomial function p such that, for any $w \in \{0,1\}^*$,

$$\begin{aligned} w \in A &\iff (\exists u, \ell(u) = p(\ell(w)))\, \langle w, u\rangle \in B \\ &\iff (\exists \text{ a unique } u, \ell(u) = p(\ell(w)))\, \langle w, u\rangle \in B. \end{aligned}$$

(c) A function $\phi : \{0,1\}^* \to \mathbb{N}$ is in $\#P$ if and only if there exist a set $B \in P$ and a polynomial function p such that, for any $w \in \{0,1\}^*$,

$$\phi(w) = \|\{u \in \{0,1\}^* : \ell(u) = p(\ell(w)), \langle w, u\rangle \in B\}\|.$$

It is known that $P \subseteq UP \subseteq NP \subseteq P^{\#P}$ and $FP \subseteq \#P \subseteq FP^{\#P}$. Whether any of these inclusive relations is proper is not known and is a major open question in complexity theory.

For any of the above function classes $\mathcal{C}$, we write $\mathcal{C}_1$ to denote the class of functions $\phi : \{0\}^* \to \mathbb{N}$ that are in $\mathcal{C}$. These classes also satisfy the relation $FP_1 \subseteq \#P_1 \subseteq FP_1^{\#P}$, and whether any of the relations is a proper inclusion is also open.

2.2 Complexity of real functions and two-dimensional sets

The basic objects in the Turing machine-based continuous computation are dyadic rationals $\mathbb{D} = \{m/2^n : m \in \mathbb{Z}, n \in \mathbb{N}\}$. Each dyadic rational d has infinitely many binary representations, with arbitrarily many trailing zeros. For each $n \in \mathbb{N}$, we let $\mathbb{D}_n$ denote the class of dyadic rationals which have a binary representation of at most n bits to the right of the binary point; that is, $\mathbb{D}_n = \{m/2^n : m \in \mathbb{Z}\}$.

A real number has a few basic representations. The most basic one is the *Cauchy function representation*. We say a function $\phi : \mathbb{N} \to \mathbb{D}$ *binary converges* to a real number x, or is *a Cauchy function representation of* x, if (i) for all $n \geq 0$, $\phi(n) \in \mathbb{D}_n$, and (ii) for all $n \geq 0$, $|x - \phi(n)| \leq 2^{-n}$. A real number x may have many Cauchy function representations. However, there is a unique function $\phi_x : \mathbb{N} \to \mathbb{D}$ that binary converges to x and satisfies the condition $x - 2^{-n} < \phi_x(n) \leq x$ for all $n \geq 0$. We call this function ϕ_x the *standard Cauchy function* for x. We say a real number x is *computable* if it has a computable Cauchy function representation. A real number x is *polynomial-time computable* (or, simply, *P-computable*) if it has a Cauchy function representation $\phi : \{0\}^* \to \mathbb{D}$ in FP.[5] We write $P_\mathbb{R}$ to denote the set of all P-computable real numbers. Similarly, we write $\#P_\mathbb{R}$ (or, $P_\mathbb{R}^{\#P}$) to denote the set of real numbers which have a Cauchy function representation $\phi : \{0\}^* \to \mathbb{D}$ such that the function $\phi'(0^n) = \phi(0^n) \cdot 2^n$ is in $\#P$ (or, respectively, in $FP^{\#P}$). We note that the relation between $P_\mathbb{R}$ and $\#P_\mathbb{R}$ depends on that between FP_1 and $\#P_1$: $FP_1 = \#P_1$ if and only if $P_\mathbb{R} = \#P_\mathbb{R}$ (see Theorem 5.32 of Ko [12]).

To compute a real-valued function $f : \mathbb{R} \to \mathbb{R}$, we use oracle TMs as the computational model. We say an oracle TM M *computes* a function $f : \mathbb{R} \to \mathbb{R}$ if, for a given oracle ϕ that binary converges to a real number x and for a given input $n > 0$, $M^\phi(n)$ halts and outputs a dyadic rational e such that $|e - f(x)| \leq 2^{-n}$. We say a function $f : \mathbb{R} \to \mathbb{R}$ is *polynomial-time computable* (or, simply, *P-computable*) if there exists a polynomial-time oracle TM that computes f.

We write $\mathbf{z}$, Z or $\langle x, y \rangle$, where $x, y \in \mathbb{R}$, to denote a point in the two-dimensional plane $\mathbb{R}^2$. For any two points $\mathbf{z}_1 = \langle x_1, y_1 \rangle$ and $\mathbf{z}_2 = \langle x_2, y_2 \rangle$ in $\mathbb{R}^2$, we write $|\mathbf{z}_1 - \mathbf{z}_2|$ to denote the distance $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ between them. For any point $\mathbf{x} \in \mathbb{R}^2$ and a closed set $A \subseteq \mathbb{R}^2$, we write $\mathrm{dist}(\mathbf{x}, A) = \mathrm{dist}(A, \mathbf{x}) = \min\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in A\}$.

The notions of computable and polynomial-time computable real functions can be extended naturally to functions $f : \mathbb{R} \to \mathbb{R}^2$ and functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. In particular, we say a Jordan curve (simple, closed curve) Γ in $\mathbb{R}^2$ is *polynomial-time computable* if there exists a P-computable function $f : [0, 1] \to \mathbb{R}^2$ such that the range of f is Γ, f is one-to-one on $[0, 1)$ and $f(0) = f(1)$. For any set $S \subseteq \mathbb{R}^2$, let

[5] Note that the input integers n to ϕ are written in the form of the unary representation 0^n.

∂S denote the boundary of S, i.e., the set of all points $\mathbf{z} \in \mathbb{R}^2$ such that any neighborhood $N(\mathbf{z};\epsilon)$ of $\mathbf{z}$ contains points in S and points not in S. We say a bounded open set $S \subseteq \mathbb{R}^2$ is a *Jordan domain* if its boundary ∂S is a Jordan curve, and say it is *P-computable* if ∂S is a P-computable Jordan curve.

For any set $S \subseteq \mathbb{R}^2$, let χ_S denote the characteristic function of S; i.e., $\chi_S(\mathbf{x}) = 1$ if $\mathbf{x} \in S$, and $\chi_S(\mathbf{x}) = 0$ otherwise. Intuitively, S is computable (or, polynomial-time computable) if the function χ_S is computable (or, respectively, polynomial-time computable). Since χ_S is discontinuous at the points on ∂S, the definition based on this concept is too strict. That is, suppose that we define a set S to be computable if there is an oracle TM computing χ_S; then, only two trivial sets, $\mathbb{R}^2$ and $\emptyset$, are polynomial-time computable. Chou and Ko [3] considered two ways to relax the computability requirements of this concept, and defined the notions of polynomial-time approximable and polynomial-time recognizable sets.

A set $S \subseteq \mathbb{R}^2$ is called *polynomial-time recognizable* (or, simply, *P-recognizable*) if there exists a polynomial-time oracle TM M that, when given two oracles ϕ_1, ϕ_2 and an input $n > 0$ (written in its unary representation 0^n), computes $\chi_S(\mathbf{z})$ whenever $\langle \phi_1, \phi_2 \rangle$ represents a point $\mathbf{z}$ in $\mathbb{R}^2$ having a distance greater than 2^{-n} from the boundary ∂S; i.e, the error set

$$E_M(n) = \{\mathbf{z} \in \mathbb{R}^2 \mid (\exists \langle \phi_1, \phi_2 \rangle \text{ representing } \mathbf{z})\, M^{\phi_1,\phi_2}(n) \neq \chi_S(\mathbf{z})\} \qquad (1)$$

of M on input n is a subset of $\{\mathbf{z} \in \mathbb{R}^2 \mid \mathrm{dist}(\mathbf{z}, \partial S) \leq 2^{-n}\}$.

A set $S \subseteq \mathbb{R}^2$ is called *strongly recursively recognizable*, (or, *Strongly P-recognizable*) if it is recursively recognizable (or, respectively, P-recognizable) by an oracle TM M such that the error set $E_M(n)$ is also contained in $\mathbb{R}^2 - S$ (i.e., errors only occur when the oracles representing a point outside S, and has distance $\leq 2^{-n}$ from the boundary).

A set $S \subseteq \mathbb{R}^2$ is called *polynomial-time approximable* (or, simply, *P-approximable*) if there exists a polynomial-time oracle TM M that, when given two oracles ϕ_1, ϕ_2 representing a point $\mathbf{z} \in \mathbb{R}$, and an input 0^n, computes $\chi_S(\mathbf{z})$ with possible errors such that the Lebesgue measure of the error set $E_M(n)$, defined in (1) above, is bounded by 2^{-n}.

For any set $S \subseteq \mathbb{R}^2$, we let $CH(S)$ be the convex hull of S; that is,

$$CH(S) = \{\mathbf{z} \in \mathbb{R}^2 \mid (\exists \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in S)\, (\exists r_1, r_2, r_3 \in [0,1]) \sum_{i=1}^{3} r_i = 1, \mathbf{z} = \sum_{i=1}^{3} r_i \mathbf{z}_i\}.$$

3 Convex hull of a P-recognizable set

P-recognizability is the most general concept of polynomial-time computability for two-dimensional sets, but some of the important properties of a set are not retained in this formulation. For instance, Chou and Ko [6] pointed out that the distance function $\delta_S(\mathbf{z}) = \mathrm{dist}(\mathbf{z}, \partial S)$ is not necessarily computable even if S is P-recognizable. It is not hard to see that this is also true for the notion of convex hulls. As a simple example, suppose S consists of four corners of a square $[0, x] \times [0, x]$

where x is a noncomputable real number. Then, S is P-recognizable since all its points are on the boundary ∂S and so a trivial oracle TM M that always outputs 0 computes χ_S correctly for all points away from the boundary. On the other hand, we note that $CH(S)$ is exactly the square $R = [0, x] \times [0, x]$. It is not hard to see that R is recursively recognizable if and only if x is a computable real number.

In the following, we show that, even if S is a Jordan domain and is P-recognizable, its convex hull $CH(S)$ is not necessarily recursively recognizable.

Theorem 3.1 *There exists a P-recognizable Jordan domain S of which the convex hull $CH(S)$ is not recursively recognizable.*

Proof. Let $K \subseteq \mathbb{N}$ be an r.e., nonrecursive set of integers. Then, there exists a TM M_K that enumerates the integers in K. That is, M_K prints, on input 0, integers on its output tape one by one such that (i) it prints only integers in K, and (ii) every integer in K is eventually printed. For $n \in K$, let $t(n)$ be the number of moves M_K takes to print integer n on input 0. Without loss of generality, we assume that $t(n) \geq 2n + 1$.

Let O denote the origin $\langle 0, 0\rangle$ of $\mathbb{R}^2$ and C denote the unit circle, i.e., the circle with center O and radius 1. For any $n > 0$, let $a_n = 1/4 - 2^{-(n+1)}$, $Z_n = \langle \cos(2\pi a_n), \sin(2\pi a_n)\rangle$, and C_n be the chord of C connecting the points Z_n and Z_{n+1}.

We now define a function $f : [0, 1] \to \mathbb{R}^2$ whose image is a Jordan curve Γ. On $[1/4, 1]$, the image of f is the circle C on the second, third, and fourth quadrants; i.e., $f(t) = \langle \cos(2t\pi), \sin(2t\pi)\rangle$, for $t \in [1/4, 1]$. Next, for each $n > 0$, if $n \notin K$, then f is linear on $[a_n, a_{n+1}]$, with $f(a_n) = Z_n$ and $f(a_{n+1}) = Z_{n+1}$; i.e., f maps $[a_n, a_{n+1}]$ linearly to the chord C_n. If $n > 0$ and $n \in K$, then f maps $[a_n, a_{n+1}]$ to the chord C_n with a bump in the middle, where the bump has width $2^{-t(n)}$ and height $h_n = 1 - \cos(2^{-(n+2)}\pi)$. To be more precise, let X'_n be the middle point of the chord C_n, and X_n the intersection point of the circle C and the halfline $\overrightarrow{OX}$. Define P_n and Q_n to be the two points on C_n with distance $2^{-t(n)-1}$ from X'_n (with P_n closer to Z_n and Q_n closer to Z_{n+1}).[6] The function f is piecewise linear on $[a_n, a_{n+1}]$ with $f(a_n) = Z_n$, $f((a_n + a_{n+1})/2 - 2^{-t(n)-n-3}) = P_n$, $f((a_n + a_{n+1})/2 = X_n$, $f((a_n + a_{n+1})/2 + 2^{-t(n)-n-3}) = Q_n$, and $f(a_{n+1}) = Z_{n+1}$. (Figure 1 shows the curve Γ between Z_n and Z_{n+1}.) This completes the definition of function f. Note that f is a continuous function but is not computable.

Let S be the interior of the Jordan curve Γ. We claim that S is P-recognizable. First, it is easy to see that the set S_0 that is enclosed by the curve $f[1/4, 1]$ plus all chords C_n, for $n > 0$, is P-recognizable. Next let B_k be the area enclosed by the chord C_n and the circle C from Z_n to Z_{n+1}, and let $S_k = S \cap B_k$. If $k \notin K$, then $S_k = \emptyset$; and if $k \in K$, then S_k is a small bump of width $2^{-t(n)}$ and height h_n. Now, consider the following algorithm for the membership problem of S:

Oracles: $\langle \phi_1, \phi_2\rangle$ representing a point $\mathbf{z} \in \mathbb{R}^2$.
Input: $n > 0$.

[6] Note that $t(n) \geq 2n + 1$ implies that $2^{-t(n)-1} \leq 2^{-2n-2}$, and the distance between Z_n and X'_n is $\sin(2^{-n-2}\pi) > 2^{-n-2}$. Therefore, P_n and Q_n are between Z_n and Z_{n+1}.

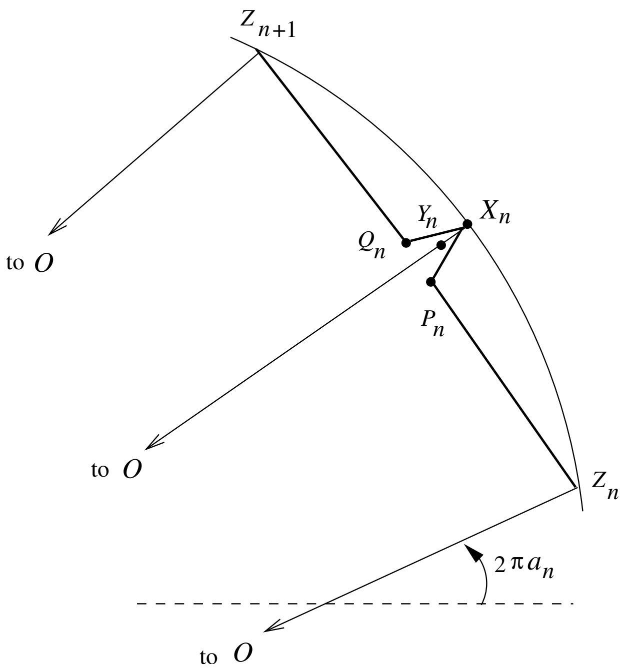

Fig. 1. The curve ∂S between Z_n and Z_{n+1}

(1) Ask the oracles to get a dyadic point $\mathbf{w} \in \mathbb{D}^2_{n+1}$ with $|\mathbf{w} - \mathbf{z}| < 2^{-(n+1)}$.
(2) If $\mathbf{w} \in S_0$, then answer YES;
(3) Else if $\mathbf{w} \notin B_k$ for any $k \leq n$, then answer NO;
(4) Else if $\mathbf{w} \in B_k$ for some $k \leq n$, then simulate TM M_K for n moves, and answer YES if and only if M_K prints k within n moves and $\mathbf{w} \in S_k$.

To see that the above algorithm solves the membership problem of S correctly, assume that $\mathbf{z}$ is a point in $\mathbb{R}^2$ with $\text{dist}(\mathbf{z}, \Gamma) > 2^{-n}$. Then, if $\mathbf{z} \in S_0$ or if $\mathbf{z}$ lies outside C, then the answer given by the algorithm is correct. Next, suppose $\mathbf{z} \in B_k$ for some $k > 0$. If $k \notin K$, or if $k \in K$ and $t(k) \leq n$, then again the answer is correct. Finally, suppose $\mathbf{z} \in B_k$ with $k \in K$ and $t(k) > n$. Then, S_k is a small bump of width $2^{-t(k)} < 2^{-n}$, and so all points in S_k have distance at most $2^{-(n+1)}$ from the boundary Γ. Thus, the answer NO is correct for $\mathbf{z}$ if it has distance $> 2^{-n}$ from Γ.

Next, we verify that this algorithm works in polynomial time. It is apparent that steps (1)—(3) and the first half of step (4) can be done in time polynomial in n. For the second half of step (4), we note that if $t(k) > n$, then we can simulate M_k for n steps and answer NO. Otherwise, if $t(k) \leq n$, then we can calculate $t(k)$ in $O(n)$ moves, and compute points X_n, P_n, Q_n correctly within error $2^{-(n+1)}$ in time polynomial in n. From these points, we can then determine whether $\mathbf{w} \in S_k$ if $\mathbf{w}$ has distance $> 2^{-(n+1)}$ from the line segments $\overline{P_n X_n}$, $\overline{X_n Q_n}$. This completes the proof that S is P-recognizable.

Now, let us consider the convex hull $CH(S)$ of set S. For each $n > 0$, let $T_n = CH(S) \cap B_n$. Note that the curve Γ lies completely within C and it includes all points Z_n. Therefore, T_n depends only on the curve Γ between Z_n and Z_{n+1}.

That is, for $n \notin K$, $T_n = \emptyset$; and for $n \in K$, T_n is equal to $\Delta Z_n X_n Z_{n+1}$, the triangle with the vertices Z_n, X_n and Z_{n+1}. Now, suppose that $CH(S)$ is recursively recognizable. Then, we can determine whether $n \in K$ as follows:

Let Y_n be the middle point in $\overline{X'_n X_n}$, and determine whether Y_n is inside $CH(S)$ with error $\leq 2^{-2n-6}$. Answer $n \in K$ if and only if $Y_n \in CH(S)$.

Note that $h_n = 1 - \cos(2^{-(n+2)}\pi) \geq 2^{-2n-4}$, and the length of C_n is $2\sin(2^{-(n+2)}\pi) \geq 2^{-n-2}$.[7] Now, it is not hard to see that the distance between Y_n and the boundary of $CH(S)$ is greater than $h_n/4$, no matter whether $n \in K$ (or, equivalently, whether $Y_n \in CH(S)$). Thus, the above algorithm determines whether $n \in K$ correctly. This is a contradiction to the assumption that K is not a recursive set. □

4 Convex hull of a P-computable Jordan domain

In this section, we consider the complexity of convex hulls of P-computable Jordan domains. In order to characterize the complexity of convex hulls, we need to extend the notion of P-recognizable sets to NP-recognizable sets.

Definition 4.1 (a) A set $T \subseteq \mathbb{R}^2$ is *NP-recognizable* if there exists a polynomial-time nondeterministic oracle TM M such that, on oracles $\langle \phi_1, \phi_2 \rangle$ representing a point $\mathbf{z} \in \mathbb{R}^2$, and on input $n > 0$,

(i) For $\mathbf{z} \in T$ with $\mathrm{dist}(\mathbf{z}, \partial T) > 2^{-n}$, $M^{\phi_1,\phi_2}(n)$ contains at least one accepting path, and

(ii) For $\mathbf{z} \notin T$ with $\mathrm{dist}(\mathbf{z}, \partial T) > 2^{-n}$, $M^{\phi_1,\phi_2}(n)$ has no accepting paths.

(b) A set $T \subseteq \mathbb{R}^2$ is *strongly NP-recognizable* if it is NP-recognizable and the nondeterministic oracle TM M also satisfies the following stronger condition

(i′) For all $\mathbf{z} \in T$, $M^{\phi_1,\phi_2}(n)$ contains at least one accepting path.

Theorem 4.2 *Assume that $S \subseteq [0,1]^2$ is a Jordan domain whose boundary ∂S is P-computable. Then, its convex hull $CH(S)$ is strongly NP-recognizable.*

Proof. Let $S^{c\ell}$ denote the closure of S; i.e., $S^{c\ell} = S \cup \partial S$. We note that, as S is a Jordan domain, $CH(S^{c\ell}) = CH(S) \cup CH(S)^{c\ell}$. Since the notion of P- and NP-recognizable sets allows the machine to have errors near the boundary of the set, $CH(S)$ and $CH(S^{c\ell})$ have the same complexity as far as we are only concerned with these complexity notions. So, in the following, we will work directly with the convex hull $CH(S^{c\ell})$ of the closed set $S^{c\ell}$.

We note that a point $\mathbf{z}$ belongs to $CH(S^{c\ell})$ if and only if there exist three points on the boundary ∂S such that $\mathbf{z}$ lies in the triangle D formed by these three points. The following algorithm for the membership problem of $CH(S)$ is based on this idea.

[7] By the Taylor expansion of the functions cos and sin, we know that for small t, $1 - \cos t \geq t^2/2 - t^4/24 \geq t^2/4$, and $2\sin t \geq 2(t - t^3/6) \geq t$.

Assume that the function $f : [0,1] \to \mathbb{R}^2$ represents the boundary ∂S, and that f is computable in time $p(n)$ for some polynomial p.

Oracles: $\langle \phi_1, \phi_2 \rangle$ representing a point $\mathbf{z} \in \mathbb{R}^2$.
Input: $n > 0$.

(1) Ask oracles $\langle \phi_1, \phi_2 \rangle$ to get a dyadic point $\mathbf{w} \in \mathbb{D}_{n+3}{}^2$ such that $|\mathbf{w} - \mathbf{z}| \leq 2^{-(n+2)}$.
(2) Nondeterministically guess three dyadic numbers $d_1, d_2, d_3 \in \mathbb{D}_{p(n+3)}$.
(3) Compute three dyadic points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{D}_{n+4}{}^2$ such that $|\mathbf{x}_i - f(d_i)| \leq 2^{-(n+3)}$ for $i = 1, 2, 3$.
(4) Let D be the triangle whose three vertices are $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$. Accept $\mathbf{z}$ if $\mathbf{w}$ is inside D or has distance $\leq 2^{-(n+1)}$ from the boundary ∂D of D.

It is clear that the above algorithm works in polynomial time. To see that the above algorithm strongly recognizes $CH(S^{c\ell})$, we first assume that $\mathbf{z} \in CH(S^{c\ell})$. Then, there must be three numbers $t_1, t_2, t_3 \in [0,1]$ such that $\mathbf{z}$ lies in the triangle D_0 formed by three vertices $f(t_1), f(t_2)$ and $f(t_3)$.

Suppose that, for each $i = 1, 2, 3$, we have a dyadic number $d_i \in \mathbb{D}_{p(n+4)}$ and dyadic point $\mathbf{x}_i \in \mathbb{D}_{n+4}{}^2$ such that $|d_i - t_i| \leq 2^{-p(n+3)}$, and $|\mathbf{x}_i - f(d_i)| \leq 2^{-(n+3)}$. Then, $|\mathbf{x}_i - f(t_i)| \leq 2^{-(n+2)}$. Let D be the triangle with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ as the three vertices. Then, the Hausdorff distance between D_0 and D is $\leq 2^{-(n+2)}$. Therefore, $\mathbf{z}$ either lies inside D or has distance $\leq 2^{-(n+2)}$ from ∂Q. It follows that $\mathbf{w}$ either lies inside D or has distance $\leq 2^{-(n+1)}$ from ∂Q. Therefore, the computation path of the algorithm that guesses the numbers d_1, d_2, d_3 will accept $\mathbf{z}$.

Conversely, assume that the above algorithm accepts $\mathbf{z}$, with the guesses $d_1, d_2, d_3 \in \mathbb{D}_{p(n+3)}$. Then, the algorithm found a triangle D such that $\mathbf{w}$ is either inside D or has distance $\leq 2^{-(n+1)}$ from ∂D. Let D_1 be the triangle with the three vertices $f(d_1), f(d_2)$ and $f(d_3)$. Then, the Hausdorff distance between D and D_1 is $\leq 2^{-(n+3)}$. It follows that $\mathbf{w}$ is either inside D_1 or within distance $2^{-(n+1)} + 2^{-(n+3)}$ from ∂D_1. Since $|\mathbf{w} - \mathbf{z}| \leq 2^{-(n+2)}$, and since $D_1 \subseteq CH(S^{c\ell})$, the point $\mathbf{z}$ is either inside $CH(S^{c\ell})$ or within distance 2^{-n} from the boundary of $CH(S^{c\ell})$. This shows that the acceptance of the algorithm is correct. □

Corollary 4.3 *Assume that $S \subseteq [0,1]^2$ is strongly P-recognizable. Then, its convex hull $CH(S)$ is strongly NP-recognizable.*

Proof. Assume that an oracle TM M strongly P-recognizes S in time $p(n)$. We modify the algorithm of Theorem 4.2 by replacing steps (2) and (3) with

(2′) Guess three points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{D}^2_{n+3}$, and verify that $M^{\mathbf{x}_i}(n+3) = 1$ for $i = 1, 2, 3$,

where $M^{\mathbf{x}_i}(n)$ denotes the computation of M on input n with the standard Cauchy functions of $\mathbf{x}_i$ as the oracles. Then, this new nondeterministic oracle TM strongly accepts $CH(S^c)$. □

Next, we show that the result of strong NP-recognizability of the convex hulls cannot be improved to P-recognizability, unless $P = NP$.

Lemma 4.4 *For any set $A \in NP$, there exist a P-computable Jordan domain S, a P-computable (discrete) function $g : \{0,1\}^* \to \mathbb{D}$, and a polynomial function q, such that, for any $w \in \{0,1\}^*$,*

(i) The distance between $g(w)$ and the boundary of $CH(S)$ is at least $2^{-q(\ell(w))}$, and

(ii) $w \in A$ if and only if $g(w) \in CH(S)$.

Proof. Let $A \in NP$. From Proposition 2.1(a), there exist a polynomial function p and a set $B \in P$ such that, for all $w \in \{0,1\}^*$,

$$w \in A \iff (\exists u, \ell(u) = p(\ell(w)))\langle w, u\rangle \in B.$$

For any $w \in \{0,1\}^*$, we let i_w be the integer between 0 and $2^{\ell(w)} - 1$ whose $\ell(w)$-bit binary representation (with possible leading zeroes) is equal to w. Also let w' denote the successor of w in the lexicographic ordering. Now, suppose $\ell(w) = n > 0$, we define a dyadic rational number in $[0, 1/4]$: $x_w = (1 - 2^{-(n-1)} + i_w \cdot 2^{-2n})/4$, and an interval: $I_w = [x_w, x_{w'}]$. Note that I_w has length $2^{-2\ell(w)-2}$.

Next, for each $u \in \{0,1\}^{p(n)}$, we define two dyadic rationals and two subintervals of I_w as follows:

$$\begin{aligned}
x_{w,u} &= x_w + 2^{-2n-4} + i_u \cdot 2^{-p(n)-2n-4},\\
x'_{w,u} &= x_w + 2^{-2n-3} + i_u \cdot 2^{-p(n)-2n-4} = x_{w,u} + 2^{-2n-4},\\
I_{w,u} &= [x_{w,u}, x_{w,u} + 2^{-p(n)-2n-4}],\\
I'_{w,u} &= [x'_{w,u}, x'_{w,u} + 2^{-p(n)-2n-4}].
\end{aligned}$$

Now, we describe the boundary ∂S of the desired Jordan domain S. Let O be the origin, and C the unit circle with center O and radius 1. For each $w \in \{0,1\}^*$ of length n, let $Z_w = \langle \cos(2\pi x_w), \sin(2\pi x_w)\rangle$, and C_w the chord connecting Z_w and $Z_{w'}$. Then, length of C_w is equal to $2\sin(2^{-2n-2}\pi)$. We denote it by $leng(C_w)$. Let X_w be the middle point on the arc of C between Z_w and $Z_{w'}$, and h_n be the distance between X_w and the chord C_w; that is, $h_n = 1 - \cos(2^{-2n-2}\pi)$. Let B_w denote the area between the chord C_w and the arc of C from Z_w through X_w to $Z_{w'}$.

We now divide each chord C_w into four line segments of equal length, and further divide each of the two middle segments into $2^{p(n)}$ subsegments, each corresponding to a string $u \in \{0,1\}^{p(n)}$. That is, let V_w, V'_w, and V''_w be the points on C_w of distance $(1/4)leng(C_w)$, $(1/2)leng(C_w)$, and $(3/4)leng(C_w)$ from Z_w, respectively. Also let $P_{w,u}$ be the point on C_w of distance $(i_u \cdot 2^{-p(n)-2} \cdot leng(C_w))$ from V_w, and $P'_{w,u}$ the point on C_w of distance $(i_u \cdot 2^{-p(n)-2} \cdot leng(C_w))$ from V'_w. Finally, let $Q_{w,u}$ be the point in B_w that is of equal distance from $P_{w,u}$ and $P_{w,u'}$ and has distance $h_n/2$ from the chord C_w, and $Q'_{w,u}$ the point in B_w that is of equal distance from $P'_{w,u}$ and $P'_{w,u'}$ and has distance $h_n/2$ from the chord C_w (see Figure 2).

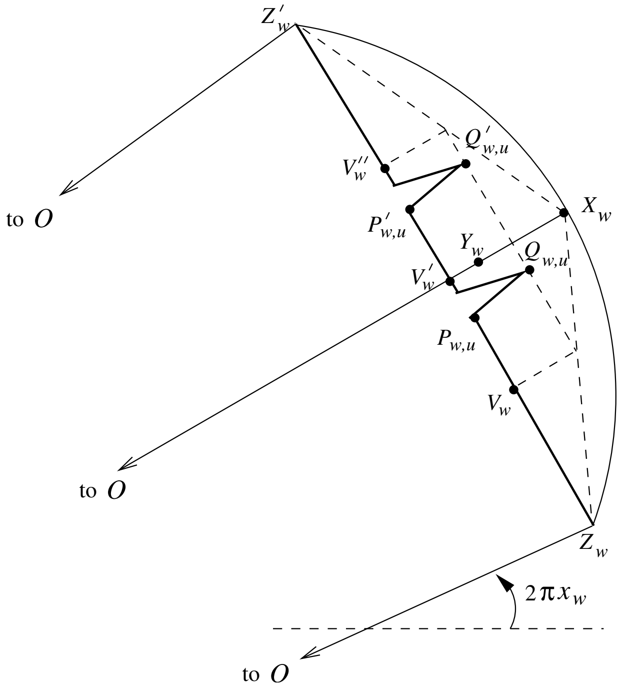

Fig. 2. The curve ∂S around C_w

Now, we are ready to define the function $f : [0,1]\to\mathbb{R}^2$ that computes the boundary ∂S of the desired Jordan domain S. First, f maps $[1/4, 1]$ to the unit circle C on the second, third, and fourth quadrants; i.e., $f(t) = \langle \cos(2t\pi), \sin(2t\pi)\rangle$, for $t \in [1/4, 1]$. Next, on each interval $I_w = [x_w, x_{w'}]$, f maps $[x_w, x_w + 2^{-2n-4}]$ linearly to the line segment $\overline{Z_wV_w}$, and maps $[x_w + 3 \cdot 2^{-2n-4}, x_{w'}]$ linearly to the line segment $\overline{V''_wZ_{w'}}$. For each $u \in \{0,1\}^{p(n)}$, if $\langle w, u\rangle \notin B$, then f maps $I_{w,u}$ linearly to the line segment $\overline{P_{w,u}P_{w,u'}}$, and maps $I'_{w,u}$ linearly to the line segment $\overline{P'_{w,u}P'_{w,u'}}$. If $\langle w, u\rangle \in B$, then f maps $I_{w,u}$ piecewise linearly to two line segments: $\overline{P_{w,u}Q_{w,u}}$ and $\overline{Q_{w,u}P_{w,u'}}$, and maps $I'_{w,u}$ piecewise linearly to two line segments $\overline{P'_{w,u}Q'_{w,u}}$ and $\overline{Q'_{w,u}P'_{w,u'}}$. This completes the definition of f. Finally, we let $g(w)$ be the point Y_w between O and X_w that has distance $3 \cdot h_n/4$ from X_w. Figure 2 shows the curve ∂S in the area B_w.

It is not hard to see that the function f and g are polynomial-time computable. We omit the details of the proof.

We now check that the domain S satisfies the required conditions. As we argued in the proof of Theorem 3.1, our design of the curve ∂S guarantees that the part of the convex hull $CH(S)$ within B_w depends only on the curve ∂S between Z_w and $Z_{w'}$. More precisely, if $w \notin A$, then $CH(S) \cap B_w = \emptyset$, and $Y_w \notin CH(S)$. On the other hand, if $w \in A$, then $S \cap B_w$ contains at least two bumps which lie to the two sides of Y_w, and so $Y_w \in CH(S)$. Furthermore, we claim that, no matter whether $Y_w \in CH(S)$, the distance between Y_w and the boundary of $CH(S)$ is greater than $2^{-p(n)-4n-5}$.

For the case of $Y_w \notin CH(S)$, we know that the chord C_w is part of the boundary of $CH(S)$, and $\mathrm{dist}(Y_w, C_w) = h_n/4$. For the case of $Y_w \in CH(S)$, let us assume that ∂S passes through two points $Q_{w,u}$ and $Q'_{w,u}$. Then, the line segment $\overline{Q_{w,u}Q'_{w,u}}$ forms part of the boundary of the convex hull $CH(S)$, and Y_w has distance $h_n/4$ from this boundary. In addition, we know that both $Q_{w,u}$ and $Q'_{w,u}$ have distance at least $(2^{-p(n)-3} \cdot leng(C_w))$ away from the line $\overline{OX_w}$. It implies that Y_w has distance at least $(2^{-p(n)-3} \cdot leng(C_w))$ from other parts of the boundary of $CH(S)$. That is, no matter whether $Y_w \in CH(S)$, $\mathrm{dist}(Y_w, \partial S) \geq \min\{h_n/4, 2^{-p(n)-3} \cdot leng(C_w)\}$.

Note that $h_n = 1 - \cos(2^{-2n-2}\pi) \geq 2^{-4n-3}$, and $leng(C_w) = 2\sin(2^{-2n-2}\pi) \geq 2^{-2n-2}$. Therefore, $\mathrm{dist}(Y_w, \partial S) \geq 2^{-p(n)-4n-5}$. This completes the proof of the claim. The proof of the lemma is also complete by setting $q(n) = p(n) + 4n + 5$. □

Theorem 4.5 *Assume that $P \neq NP$. Then, there exists a Jordan domain $S \subseteq \mathbb{R}^2$ whose boundary ∂S is P-computable but whose convex hull $CH(S)$ is not P-recognizable.*

Proof. Assume that the convex hull $CH(S)$ of the set S constructed in Lemma 4.4 is P-recognizable. Then, we can determine whether $w \in A$ by asking whether $g(w)$ is in $CH(S)$, with error bound $< 2^{-q(n)}$. □

Corollary 4.6 *Assume that $P \neq NP$. Then, there exists a Jordan domain $S \subseteq \mathbb{R}^2$ which is strongly P-recognizable but whose convex hull $CH(S)$ is not P-recognizable.*

5 Areas of Convex Hulls

In this section, we consider the complexity of computing the area of the convex hull $CH(S)$ of a P-computable Jordan domain S. We first recall the results about the complexity of computing the area of a set T in the two-dimensional plane.

Proposition 5.1 *(a) If $T \subseteq [0,1]^2$ is P-approximable, then area of T is a real number in $\#P_{\mathbb{R}}$.*

(b) If $T \subseteq [0,1]^2$ is a P-recognizable Jordan domain with a rectifiable boundary, then area of T is in $\#P_{\mathbb{R}}$.

(c) If $FP_1 \neq \#P_1$, then there exists a convex set $T \subseteq [0,1]^2$ that is P-approximable but its area is not in $P_{\mathbb{R}}$.

Remarks. (1) Friedman [10] proved that the integral $\int_0^1 f$ of a P-computable function $f : [0,1] \to \mathbb{R}$ is a real number in $\#P_{\mathbb{R}}$. Parts (a) and (b) of Proposition 5.1 are due to Chou and Ko [3], in which the result of [10] was extended to the measure of two-dimensional P-approximable and P-recognizable sets.

(2) Friedman [10] also showed that, if $FP \neq \#P$, then the integral $\int_0^1 f$ of some P-computable function $f : [0,1] \to \mathbb{R}$ is not in $P_{\mathbb{R}}$. Du and Ko [8] and Chou and Ko [3] extended this result to two-dimensional, P-approximable, convex sets.

We note that a convex Jordan domain T must have a rectifiable boundary. Therefore, if the convex hull $CH(S)$ of a Jordan domain is P-recognizable, then its area is a real number in $\#P_{\mathbb{R}}$. This observation can be easily extended to NP-recognizable convex hulls. We first need to extend the notion of $\#P$-computable real numbers to $\#NP$-computable real numbers.

Definition 5.2 We define the class $\#NP$ (or, $\#\cdot NP$) [8] to be the class of functions $\phi : \{0,1\}^* \to \mathbb{N}$ with the following property: There exist a set $B \in NP$ and a polynomial function p such that, for any $w \in \{0,1\}^*$,

$$\phi(w) = \|\{u \in \{0,1\}^* : \ell(u) = p(\ell(w)), \langle w, u \rangle \in B\}\|.$$

We let $\#NP_{\mathbb{R}}$ denote the class of real numbers x which have a Cauchy function representation $\phi : \{0\}^* \to \mathbb{D}$ such that the function $\phi'(0^n) = \phi(n) \cdot 2^n$ is a function in $\#NP$.

Theorem 5.3 *Assume that S is a P-computable Jordan domain. Then, the area of $CH(S)$ is a real number in $\#NP_{\mathbb{R}}$.*

Proof. Without loss of generality, assume that $S \subseteq [0,1]^2$. Also assume that the boundary of $CH(S)$ has length bounded by a. Assume that M is a nondeterministic polynomial-time oracle TM that strongly NP-recognizes $CH(S)$, as given in Theorem 4.2. For any $n > 0$, let

$$B = \{\langle 0^n, d_1, d_2 \rangle \mid d_1, d_2, \in \mathbb{D}_n, M^{d_1,d_2}(n) \text{ accepts}\},$$

where M^{d_1,d_2} denotes the computation of the machine M using the standard Cauchy functions for d_1 and d_2 as the oracles. It is clear that $B \in NP$. Furthermore, the function

$$\phi(0^n) = \|\{\langle d_1, d_2 \rangle \mid d_1, d_2 \in \mathbb{D}_n, \langle 0^n, d_1, d_2 \rangle \in B\}\|$$

is a function in $\#NP$ such that the function $\psi(0^n) = \phi(0^n) \cdot 2^{-2n}$ converges to the area of $CH(S)$ with error $|\psi(0^n) - area(CH(S))| \leq a \cdot 2^{-2n+2}$. $\square$

Next, we study whether $CH(S)$ is actually a real number in $\#P_{\mathbb{R}}$. For this question, we need to review more results about the relations between counting complexity classes in discrete complexity theory.

In his celebrated paper about counting complexity classes, Toda [18] showed that $PP^{PH} \subseteq P^{\#P[1]}$; that is, if a set is computable in probabilistic polynomial time relative to a set in the polynomial-time hierarchy, then it is computable in polynomial-time with a single query to an oracle function in $\#P$. [9] Toda and Watanabe [19] further extended this result to the function classes and showed that $\#P^{PH} \subseteq FP^{\#P[1]}$. Since $\#NP$ is a subclass of $\#P^{PH}$, the following result is immediate.

[8] In the original paper of Valiant [20], the notation $\#NP$ was defined to mean the class $\#P^{NP}$. Hemaspaandra and Vollmer [11] pointed out that, in view of the characterization of $\#P$ of Proposition 2.1(c), it appears to be more approapriate to define $\#NP$ to mean the class we defined here, and proposed, in a general framework, the notation $\# \cdot NP$ for this class. Here, we use $\#NP$ for its simplicity.

[9] Here, PP denotes the class of sets accepted by polynomial-time probabilistic TMs with accepting probability greater than 1/2, and PH denotes the polynomial-time hierarchy, of which NP is the first level. For more details, see Du and Ko [9].

Proposition 5.4 $\#NP \subseteq FP^{\#P[1]}$.

Combining Propositions 5.1 and 5.4, we obtain the following results about the area of $CH(S)$.

Corollary 5.5 *Assume that $S \subseteq \mathbb{R}^2$ is a P-computable Jordan domain. Then, the area of $CH(S)$ is a real number in $P_{\mathbb{R}}^{\#P}$.*

Corollary 5.6 *The following are equivalent:*
(a) For any P-computable Jordan domain $S \subseteq \mathbb{R}^2$, the area of $CH(S)$ is in $P_{\mathbb{R}}$.
(b) $FP_1 = \#P_1$.

Corollary 5.5 leaves it open whether the area of $CH(S)$ is actually in $\#P_{\mathbb{R}}$. This question is clearly related to the question of whether the discrete classes $\#P$ and $\#NP$ are equal. The following nice characterization of this question is due to Hemaspaandra and Volmer [11].

Proposition 5.7 *$NP = UP$ if and only if $\#P = \#NP$.*

Corollary 5.8 *If $UP = NP$, then area of the convex hull $CH(S)$ of a P-computable Jordan domain S is in $\#P_{\mathbb{R}}$.*

Whether the converse of the above holds remains open. We note that Proposition 5.7 implies that if $UP \neq NP$ then there exists some function ψ in $\#NP$ that is not in $\#P$. However, this function ψ constructed in the proof in Hemaspaandra and Vollmer [11] is a simple, characteristic function of a set $A \in NP - UP$. It seems difficult to construct a P-computable Jordan curve S of which the area of $CH(S)$ is related to such a function ψ. It would be interesting to find out whether a stronger condition of separating some discrete classes implies that the area of $CH(S)$ is not in $\#P_{\mathbb{R}}$.

Acknowledgment

The authors would like to thank Professor Joseph Mitchell for his suggestion of studying the complexity of convex hulls, and Professor Lane Hemaspaandra for the discussion on the relation between the counting classes $\#P$ and $\#NP$ and pointing out the relevant references.

References

[1] Braverman, M., *Hyperbolic Julia sets are poly-time computable*, Proceedings of the 6th Workshop on Computability and Complexity in Analysis, Electron. Notes Theoret. Comput. Sci. **120** (2005), 17–30.

[2] Braverman, M. and M. Yampolsky, *Non-computable Julia sets*, J. Amer. Math. Soc. **19** (2006), 551–578.

[3] Chou, A. W. and K.-I. Ko, *Computational Complexity of Two-Dimensional Regions*, SIAM. J. Comput. **24** (1995), 923–947.

[4] Chou, A. W. and K.-I. Ko, *On the complexity of finding paths in a two-dimensional domain I: Shortest paths*, Math. Logic Quart. **50** (2004), 551–572.

[5] Chou, A. W. and K.-I. Ko, *On the complexity of finding paths in a two-dimensional domain II: Piecewise straight-line paths*, Proceedings of the 6th Workshop on Computability and Complexity in Analysis, Electron. Notes Theoret. Comput. Sci. **120** (2005), 45–57.

[6] Chou, A. W. and K.-I. Ko, *The computational complexity of distance functions of two-dimensional domains*, Theoret. Comput. Sci. **337** (2005), 360–369.

[7] De Berg, M., M. van Kreveld, M. Overmars, and O. Schwarzkopf, "Computational Geometry: Algorithms and Applications", 2nd Ed., Springer-Verlag, New York, 2000.

[8] Du, D. and K.-I. Ko, *Computational complexity of integration and differentiation of convex functions*, System Sci. Math. Sci. **2** (1989), 70–79.

[9] Du, D.-Z. and K.-I. Ko, "Theory of Computational Complexity", John Wiley & Sons, New York, 2000.

[10] Friedman, H., *On the computational complexity of maximization and integration*, Adv. in Math. **53** (1984), 80–98.

[11] Hemaspaandra, L. and H. Vollmer, *The Satanic notations: Counting classes beyond #P and other definitional adventure*, SIGACT News **26** (1995), 2–13.

[12] Ko, K.-I., "Complexity Theory of Real Functions", Birkhäuser, Boston, 1991.

[13] Ko, K. and F. Yu, *On the complexity of computing the logarithm and square root functions on a complex domain*, J. Complexity **23** (2007), 2–24.

[14] O'Rourke, J., "Computational Geometry in C", 2nd Ed., Cambridge University Press, Cambridge, UK, 1996.

[15] Pour-El, M. B. and J. I. Richards, "Computability in Analysis and Physics", Perspectives in Mathematical Logic, Springer, Berlin, 1989.

[16] Rettinger, R., *A fast algorithm for Julia sets of hyperbolic rational functions*, Proceedings of 6th Workshop on Computability and Complexity in Analysis, Electronic Notes in Theoret. Comput. Sci. **120** (2005), 145–157.

[17] Rettinger, R. and K. Weihrauch, *The computational complexity of some Julia sets*, Proceedings of 35th ACM Symposium on Theory of Computing, 2003, 177–185.

[18] Toda, S., *PP is as hard as the polynomial-time hierarchy*, SIAM J. Comput. **20** (1991), 865-877.

[19] Toda, S. and O. Watanabe, *Polynomial-time 1-turing reductions from #PH to #P*, Theoret. Comput. Sci. **100** (1992), 205–221.

[20] Valiant, L., *The complexity of computing the permanent*, Theoret. Comput. Sci. **8** (1979), 189–201.

[21] Weihrauch, K., "Computable Analysis," Springer, Berlin, 2000.